Basic Group Theory

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Given a group G and an action on a set X we claim that X is in bijection with the union of its disjoint orbits.

Proof. It is clear that

$$X = \cup_x Gx$$

But if x and y are in the same orbit then it is also clear that $Gx = Gy$. Thus we can pick a set of representatives such that the union is over disjoint sets.

$$X = \sqcup_{x_i} Gx_i$$

A given orbit is in bijection with the G modulo the stabiliser of a point in the orbit. Note that the stabilizer is not usually normal and so this is a set quotient.

Proof. Consider $x \in X$ then we wish to show that

$$Gx \cong G/\text{stab}(x)$$

Consider the map

$$G \to X$$

$$g \mapsto gx$$

Clearly the image is the orbit Gx. It is clear that $f(g) = f(h)$ iff $g^{-1}h$ is in the stabiliser of x. Therefore it defines a bijection from $G/\text{stab}(x)$ to Gx. This can be seen as an application of the first isomorphism theorem for sets

$$\text{Im} f \cong \text{Dom}(f)/(g \sim h \iff f(g) = f(h))$$

In particular a transitive action of a group has only a single orbit and so exhibits the set as a quotient of the group.

Remark. These facts can be combined with the isomorphism theorems of groups/ sets (both "universal algebras") to do all of math.